Q1 $\frac{10}{4}$ 2 Asma [redacted]

(i) $T: V \to W$

$\Leftarrow$ Suppose $Z(T) \in \{0_v\}$

$T(x) = T(y)$ for some $x, y \in V$

$\Rightarrow T(x) - T(y) = 0_w$

$\Rightarrow T(x-y) = 0_w \Rightarrow x-y \in Z(T)$

$\therefore x - y = 0_v \Rightarrow x = y$

$\rightarrow$ Suppose $T: V \to W$ is 1-1

let $x \in Z(T) \Rightarrow T(x) = 0_w = T(0_v)$

$\Rightarrow x = 0_v$ since T is 1-1

$\therefore Z(T) = \{0_v\}$ ✓

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(ii) Let $M = \{v_0 + d \mid d \in Z(T)\}$

Show that $M \subseteq T^{-1}(w_0)$

Suppose $x \in M \Rightarrow x = v_0 + d$ where $d \in Z(T)$.

Then, $T(x) = T(v_0 + d)$

$= T(v_0) + T(d)$

$= w_0 + 0$

$= w_0$

$\Rightarrow T(x) = w_0 = T(v_0)$ & $x \in T^{-1}(w_0)$

Now we show that $T^{-1}(w_0) \subseteq M$

Suppose $x \in T^{-1}(w_0)$, then

$T(x - v_0) = T(x) - T(v_0)$ $\qquad x, v_0 \in T^{-1}(w_0)$

$= w_0 - w_0$

$= 0$

$\Rightarrow x - v_0 \in Z(T) \Rightarrow \exists\, d \in Z(T)$ s.t. $d = x - v_0$

(i.e. $x = v_0 + d$)

$\therefore x \in M \Rightarrow T^{-1}(w_0) \subseteq M$ and $T^{-1}(w_0) = M$

✓ Good

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what is that?

~~(Bezez, 2015)~~

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Question (1).

$T : V \to W$

(i) $(\rightarrow)$ Assume that T is one-to-one.

Show that $Z(T) = \{0_V\}$.

define $Z(T) = \{ V_0 \in V \mid T(V_0) = 0_W \}$, and we already know that $T(0_V) = 0_W$;

Then $T(V_0) = T(0_V)$.

And since T is one-to-one;

Then $V_0 = 0_V$.

Therefore, $Z(T) = \{0_V\}$.

$(\leftarrow)$ Assume that $Z(T) = \{0_V\}$.

Show that T is one-to-one.

Let $V_1, V_2 \in V$ such that $T(V_1) = T(V_2)$.

Then we have:

$$T(V_1) - T(V_2) = 0_W$$

since T is a L.T $\rightarrow$ $T(V_1) + T(-V_2) = 0_W$

since T is a L.T $\rightarrow$ $T(V_1 - V_2) = 0_W$

Then, $V_1 - V_2 \in Z(T)$.

And since $Z(T) = \{0_V\}$, then $V_1 - V_2 = 0_V$.

Then, $V_1 = V_2$.

$\therefore$ Therefore, T is one-to-one.

(ii) Take $d \in V$ such that $T(d) = 0_W$.

That is $d \in Z(T)$.

Then $T(V_0 + d) = T(V_0) + T(d)$ ← since T is a L.T

$= W_0 + 0_W = W_0$.

$\therefore T(V_0 + d) = W_0$

$\therefore T^{-1}(W_0) = V_0 + d$ one direction

Therefore, $T^{-1}(W_0) = \{V_0 + d \mid d \in Z(T)\}$.

N/A

show $T^{-1}(W_0) \subseteq \{V_0 + d \mid d \in Z(T)\}$

(iii) $T(y(x)) = a_{(n)}y^{(n)} + a_{(n-1)}y^{(n-1)} + \cdots + a_1 y' + a_0 y$.

Take $y_1, y_2 \in C^n[R]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

To show that T is a linear transformation, it's enough to show $T(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 T(y_1) + \alpha_2 T(y_2)$.

$$T(\alpha_1 y_1 + \alpha_2 y_2) = a_n[\alpha_1 y_1 + \alpha_2 y_2]^{(n)} + a_{n-1}[\alpha_1 y_1 + \alpha_2 y_2]^{(n-1)} + \cdots + a_0[\alpha_1 y_1 + \alpha_2 y_2].$$

$$= (a_n \alpha_1 y_1^{(n)} + a_n \alpha_2 y_2^{(n)}) + (a_{n-1}\alpha_1 y_1^{(n-1)} + a_{n-1}\alpha_2 y_2^{(n-1)}) + \cdots + (a_0 \alpha_1 y_1 + a_0 \alpha_2 y_2).$$

$$= (a_n \alpha_1 y_1^{(n)} + a_{n-1}\alpha_1 y_1^{(n-1)} + \cdots + a_0 \alpha_1 y_1) + (a_n \alpha_2 y_2^{(n)} + a_{n-1}\alpha_2 y_2^{(n-1)} + \cdots + a_0 \alpha_2 y_2).$$

$$= \alpha_1 (a_n y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_0 y_1) + \alpha_2 (a_n y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \cdots + a_0 y_2).$$

$$= \alpha_1 T(y_1) + \alpha_2 T(y_2).$$

$\therefore$ Hence, T is a linear transformation.

Show that $T^{-1}(f(x)) = \{ d(x) + m \mid m \in Z(T) \}$.

Take $m \in C^{\infty}[R]$ such that $T(m) = 0_{C^{\infty}[R]}$.

That's $m \in Z(T)$

Then $T(d(x) + m) = T(d(x)) + T(m)$ since T is a L.T

$= f(x) + 0_{C^{\infty}[R]}$ since $T(d(x)) = f(x)$ (given).

$= f(x)$.

$\therefore T(d(x) + m) = f(x)$

$\therefore T^{-1}(f(x)) = d(x) + m$.

Therefore, $T^{-1}(f(x)) = \{ d(x) + m \mid m \in Z(T) \}$.

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Question (2).

(a) $$D = \left\{ \begin{bmatrix} a+2b & 3a+c \\ 5a+4b+c & -2a-4b \end{bmatrix} \middle| \ a,b,c \in \mathbb{R} \right\}$$

$$\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$$
(as vector space)

Fake set correspond to D:

$$D' = \{ (a+2b,\ 3a+c,\ 5a+4b+c,\ -2a-4b) \mid a,b,c \in \mathbb{R} \}$$

$$= \{ a(1,3,5,-2) + b(2,0,4,-4) + c(0,1,1,0) \mid a,b,c \in \mathbb{R} \}$$

$$= \text{span}\{ \overset{dep}{(1,3,5,-2)},\ (2,0,4,-4),\ (0,1,1,0) \}$$

$$= \text{span}\{ (2,0,4,-4),\ (0,1,1,0) \}$$

$$IN(D') = 2$$

$$D = \text{span}\left\{ \begin{bmatrix} 2 & 0 \\ 4 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$IN(D) = 2.$$

$\therefore$ Therefore, D is a subspace of $\mathbb{R}^{2\times 2}$.

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & 0 & 4 & -4 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & -6 & -6 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{6}R_2+R_3} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ 2 leaders

$\therefore$ IN = 2

(b)

$$D = \{ (a+3b)x^3 + (-2a+b)x^2 + (-a+4b)x + (2a-b) \mid a,b \in \mathbb{R} \}$$

$$P_4 \cong \mathbb{R}^4$$
(as vector space)

Fake set correspond to D :

$$D' = \{ (a+3b, -2a+b, -a+4b, 2a-b) \mid a,b \in \mathbb{R} \}$$

$$= \{ a(1,-2,-1,2) + b(3,1,4,-1) \mid a,b \in \mathbb{R} \}$$

$$= \text{span} \{ (1,-2,-1,2), (3,1,4,-1) \}$$

$\text{IN}(D') = 2$.

$$D = \text{span} \{ (x^3 - 2x^2 - x + 2), (3x^3 + x^2 + 4x - 1) \}$$

$\text{IN}(D) = 2$.

$\therefore$ Therefore, D is a subspace of P_4.

Question (3).

$T : P_4 \to P_3$

$T(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_2 - a_1 + a_0)x^2 + (2a_2 + a_0)x + (-a_2 + a_1 + 2a_0)$

(i) $T' : \mathbb{R}^4 \to \mathbb{R}^3$

$$T'(a_0, a_1, a_2, a_3) = (-a_2 + a_1 + 2a_0,\ 2a_2 + a_0,\ a_2 - a_1 + a_0).$$

$$= \{a_0(2,1,1) + a_1(1,0,-1) + a_2(-1,2,1) + a_3(0,0,0,0)\}$$

$$= \text{span}\{(2,1,1), (1,0,-1), (-1,2,1), (0,0,0)\}$$

$$= \text{span}\{(2,1,1), (1,0,-1), (-1,2,1)\}.$$

$$M' = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

(ii) $Z(T')$:

$$M'\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{5}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \end{array}\right] \xrightarrow{-2R_2} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \end{array}\right] \xrightarrow{\frac{3}{2}R_2 + R_3}$$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \end{array}\right] \rightarrow$$

$-6a_2 = 0 \rightarrow a_2 = 0$

$a_1 - 5a_2 = 0 \rightarrow a_1 = 0$

$a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 = 0 \rightarrow a_0 = 0$

a_3 is a free variable

Sol. set $= \{(0,0,0,a_3) \mid a_3 \in \mathbb{R}\}$

$= \{a_3(0,0,0,1) \mid a_3 \in \mathbb{R}\}$

$Z(T') = \text{span}\{(0,0,0,1)\}$

$\therefore$ Sol. set of T $= \text{span}\{x^3\}$.

$\therefore Z(T) = \text{span}\{x^3\}$

(iii) $\text{Range}(T') = \overset{\text{span}}{\text{Col}}(M')$

$= \text{span}\{(1,1,2), (-1,0,1), (1,2,-1)\}$

$\text{Range}(T) = \text{span}\{(x^2+x+2,\ -x^2+1,\ x^2+2x-1)\}$

(iv) Does $(-7,3,1)$ belong to $\text{Range}(T')$?

$$M' \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$$

$\underset{T'}{\text{Sol. set}} = \{(-2, -\frac{1}{2}, \frac{5}{2}, a_3) \mid a_3 \in \mathbb{R}\} \rightarrow \underset{T}{\text{Sol. set}} = \{(-2 - \frac{1}{2}x + \frac{5}{2}x^2 + a_3x^3) \mid a_3 \in \mathbb{R}\}$

Since there exist (a_0, a_1, a_2, a_3) such that $(-7,3,1) \in \text{Range}(T')$,

then there exist a polynomaial $a_0 + a_1x + a_2x^2 + a_3x^3$ such that $-7+3x+x^2 \in \text{Range}(T)$.

$\therefore$ Answer is YES.

Question (4).

$$T : P_3 \to \mathbb{R}$$

$$\begin{cases} T(x^2) = 1 \\ T(2x) = 4 \\ T(x+1) = -4 \end{cases}$$

(a) $T' : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{cases} T'(1,0,0) = 1 \\ T'(0,2,0) = 4 \\ T'(0,1,1) = -4 \end{cases}$$

$$T'(e_1) = T'(1,0,0) = 1$$

$$T'(e_2) = T'(0,1,0) = T'\left(\tfrac{1}{2}(0,2,0)\right) = \tfrac{1}{2}T'(0,2,0) = \tfrac{1}{2}(4) = 2.$$

$$T'(e_3) = T'(0,0,1) = T'\left(-\tfrac{1}{2}(0,2,0) + (0,1,1)\right) = -\tfrac{1}{2}T'(0,2,0) + T'(0,1,1)$$

$$= -\tfrac{1}{2}(4) + (-4) = -2 - 4 = -6.$$

$$\therefore M' = \begin{array}{c} \begin{array}{ccc} T'(e_1) & T'(e_2) & T'(e_3) \end{array} \\ \begin{bmatrix} 1 & 2 & -6 \end{bmatrix} \end{array}$$

(b)

$$Z(T') \rightarrow \begin{bmatrix} 1 & 2 & -6 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \end{array}\right]$$

$$\rightarrow \quad a_2 + 2a_1 - 6a_0 = 0$$

$\rightarrow \quad a_2 = -2a_1 + 6a_0$, a_1 & a_2 are free variables.

Sol. set $Z(T') = \{ (-2a_1 + 6a_0, a_1, a) \mid a_1, a_0 \in \mathbb{R} \}$

$= \{ a_1(-2,1,0) + a_0(6,0,1) \mid a_1, a_0 \in \mathbb{R} \}$

$= \operatorname{span}\{ (-2,1,0), (6,0,1) \}$

$Z(T) = \operatorname{span}\{ (-2x^2 + x), (6x^2 + 1) \}$

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(c) $H = \{ a \in P_3 \mid T(a) = \pi \}$

??

$T(a) = \pi \rightarrow T(a_2x^2 + a_1x + a_0) = \pi$

$\rightarrow T'(a_2, a_1, a_0) = \pi$

$\rightarrow M' \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \pi$

$\rightarrow \begin{bmatrix} 1 & 2 & -6 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \pi$

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$\rightarrow a_2 + 2a_1 - 6a_0 = \pi$

$\rightarrow a_2 = \pi - 2a_1 + 6a_0$, a_1 & a_2 are free variables.

$\rightarrow$ Sol. set $(T') = \{ (\pi - 2a_1 + 6a_0, a_1, a_0) \mid a_1, a_0 \in \mathbb{R} \}$

$\rightarrow$ Sol. set $(T) = \{ ((\pi - 2a_1 + 6a_0)x^2 + a_1x + a_0) \mid a_1, a_0 \in \mathbb{R} \}$

$\therefore H = \{ (\pi - 2a_1 + 6a_0)x^2 + a_1x + a_0 \mid a_1, a_0 \in \mathbb{R} \}$

note that $T(x^2) = 1$. Hence $T(\pi x^2) = \pi$.

By Question 1(ii): $T^{-1}(\pi) = \{ \pi x^2 + m \mid m \in Z(T) \}$